Tangents To Circles

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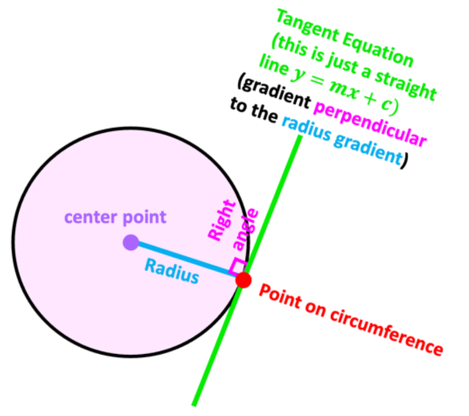
Introduction

Tangent lines and just straight lines. Make sure your straight-line graphs knowledge is good and this topic will be easy!

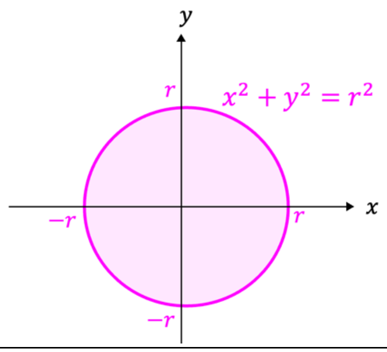
Tips:

- Draw everything out



A tangent touches a circle at one point only and meets the circle at a right angle (circle theorem)

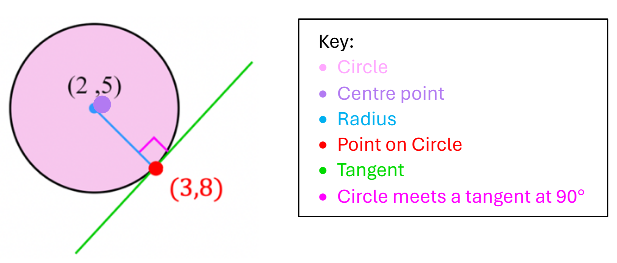
- Plug in points to find unknowns (make sure you plug them into the correct equation)
- Find areas (it is so important to have a diagram for this and locate the base and height)
- Know the equation of a circle



Easy Example - Finding The Tangent Equation Only

Find the equation of the tangent to the circle at (3,8) whose centre is (2,5). Give your answer in the form $ax + by + c = 0$

First of all let's draw all the information in



a tangent meets a radius at 90°

we can find the gradient of the radius $= \frac{8-5}{3-2} = \frac{3}{1} = 3$

tangent gradient is perpendicular so gradient is the negative reciprocal $-\frac{1}{3}$

$$y = -\frac{1}{3}x + c$$

we know the point (3, 8) lies on the tangent line so we can plug it in

$$8 = -\frac{1}{3}(3) + c$$
$$c = 8 + 1$$
$$c = 9$$
$$y = -\frac{1}{3}x + 9$$

The question asks us to leave in a certain form

This is just asking us to kill the fractions by multiplying everything by 3

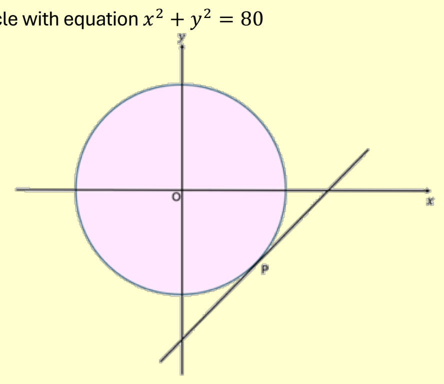
$$3y = -x + 27$$

Hence we get

$$x + 3y - 27 = 0$$

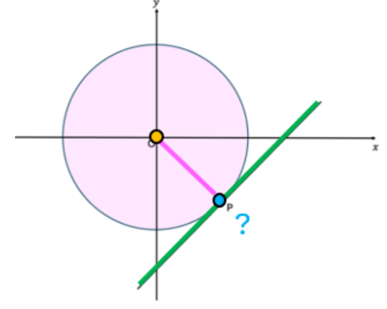
GCSE AQA June 2018 Paper 3H Q28

P is a point on the circle with equation $x^2 + y^2 = 80$



P has x coordinate 4 and is below the x axis

Work out the equation of the tangent to the circle at P giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.



Find the y coordinate of P by substituting $x = 4$ into the equation of the circle

$$4^2 + y^2 = 80$$
$$y^2 = 80 - 16$$
$$y = \sqrt{64}$$
$$y = 8, -8$$

From the diagram we know P lies below the x axis, so $y = -8$

P $(4, -8)$

Find the gradient of OP, which is perpendicular the tangent at P

$$m = \frac{-8-0}{4-0} = -2$$
$$gradient\ of\ tangent\ at\ P = \frac{1}{2}$$

Find the equation of the tangent

We know a point, P$(4, -8)$, on the tangent, as well as the gradient:

$$y = mx + c$$

Substitute in the gradient and the coordinates of P

$$-8 = \frac{1}{2} \times 4 + c$$
$$-8 = 2 + c$$
$$c = -10$$

So the equation of the tangent is $y = \frac{1}{2}x - 10$

Multiply by 2 and rearrange to get an equation with integers

$$-x + 2y + 20 = 0$$

Medium Examples - Finding Points First By Plugging In

Remember from straight line graphs that if we have an unknown we can plug in a point that lies on the line or circle to find it. Make sure it lies on the equation you're plugging into first

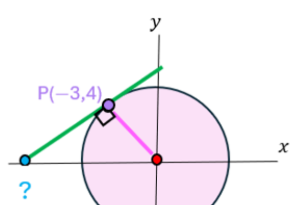
GCSE June 2023 Paper 3H Q23

A circle has equation $x^2 + y^2 = 25$

The point P with coordinates $(-3,4)$ lies on the circle

Alex says that the tangent to the circle at P crosses the x axis at the point $(-8,0)$

Is Alex correct? You must show how you get your answer.



The question is asking for the blue point

Find the gradient of the radius

$$m = \frac{4-0}{-3-0} = -\frac{4}{3}$$

We can find the gradient of the tangent

$$gradient\ of\ tangent = \text{negative reciprocal} = \frac{3}{4}$$

We now find the equation of the tangent:

We know a point, P$(-3,4)$, on the tangent as well as the gradient:

$$y = mx + c$$

Substitute in the gradient and the coordinates of P

$$4 = \frac{3}{4} \times -3 + c$$
$$4 = -\frac{9}{4} + c$$
$$c = \frac{25}{4}$$

So, the equation of the tangent is $y = \frac{3}{4}x + \frac{25}{4}$

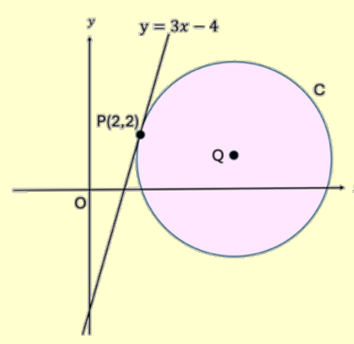
Check if the point $(-8,0)$ lies on the line $y = \frac{3}{4}x + \frac{25}{4}$

Substitute -8 into the place of x and see if you get 0

$$y = \frac{3}{4} \times -8 + \frac{25}{4} = \frac{1}{4} (\neq 0)$$

So, Alex is incorrect

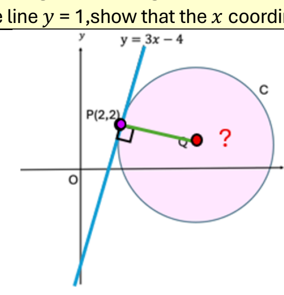
The line $y = 3x - 4$ is a tangent to the circle C, touching C at the point $P(2, 2)$, as shown in the diagram below



The point Q is the centre of C.

i.Find an equation of the straight line through P and Q.

ii.Given that Q lies on the line $y = 1$,show that the x coordinate of Q is 5



i.

Find the gradient of PQ, which is perpendicular to the tangent $y = 3x - 4$

$$gradient\ of\ radius = -\frac{1}{3} = -\frac{1}{3}$$

Find the full equation of PQ by substituting the coordinates P$(2, 2)$ into the equation of the line

$$y = mx + c$$
$$2 = -\frac{1}{3}(2) + c$$
$$c = \frac{8}{3}$$
$$y = -\frac{1}{3}x + \frac{8}{3}$$

ii.

y coordinate of Q is 1: substitute this into the above equation to find the x coordinate

$$1 = -\frac{1}{3}x + \frac{8}{3}$$
$$-\frac{1}{3}x = -\frac{5}{3}$$
$$x = 5$$

Medium Examples Continued

P is a point on a circle with centre (0,0)

The coordinates of P are $(8, -10)$

The line **L** is the tangent to the circle at the point P

L crosses the x axis at the point Q and crosses the y axis at the point R

Work out the length of RQ

Give your answer to 3 significant figures

Gradient of radius $= \frac{-10-0}{8-0} = -\frac{5}{4}$

Perpendicular tangent gradient is $\frac{4}{5}$

$$y = \frac{4}{5}x + c$$

The point $(8, -10)$ lies on the tangent

$$-10 = \frac{4}{5}(8) + c$$
$$-10 = \frac{32}{5} + c$$
$$c = -\frac{82}{5}$$
$$y = \frac{4}{5}x - \frac{82}{5}$$

When crosses the y axis, $x = 0$

$$y = \frac{4}{5}(0) - \frac{82}{5}$$
$$y = -\frac{82}{5} \text{ hence } Q\left(0, -\frac{82}{5}\right)$$

When crosses the x axis, $y = 0$

$$0 = \frac{4}{5}(x) - \frac{82}{5}$$
$$\frac{4}{5}x = \frac{82}{5}$$
$$x = \frac{41}{2} \text{ hence } R\left(\frac{41}{2}, 0\right)$$

$$\text{Length} = RQ = \sqrt{\left(\frac{41}{2}\right)^2 + \left(\frac{82}{5}\right)^2} = 26.3$$

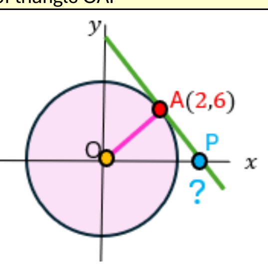
Somewhat Challenging Examples - Finding Areas

GCSE Edexcel Specimen Set 2 Paper 1H Q22

The line l is a tangent to the circle $x^2 + y^2 = 40$ at the point A.

A is the point (2,6)

The line l crosses the x axis at the point P. Work out the **area** of triangle OAP



Gradient of OA is $\frac{6-0}{2-0} = 3$

Tangent AP is perpendicular to OA: Gradient of AP $= -\frac{1}{3}$

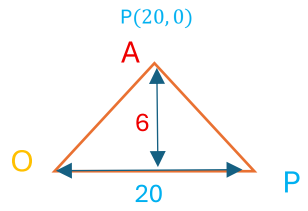
Use this to find the equation of the tangent to the circle by substituting point A$(2, 6)$ into the equation of a line

$$y = mx + c$$
$$6 = -\frac{1}{3}(2) + c$$
$$6 = -\frac{2}{3} + c$$
$$c = \frac{20}{3}$$

So the equation of the tangent at A is $y = -\frac{1}{3}x + \frac{20}{3}$

P is the x intercept of this line so sub in $y = 0$

$$0 = -\frac{1}{3}x + \frac{20}{3}$$
$$x = 20$$
$$P(20, 0)$$

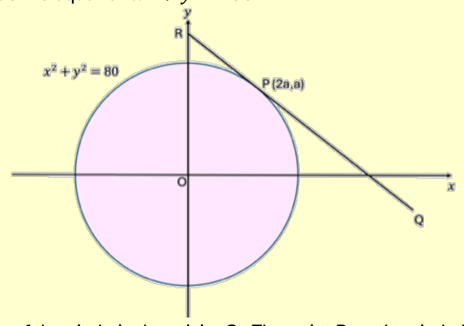


Find the area of the triangle

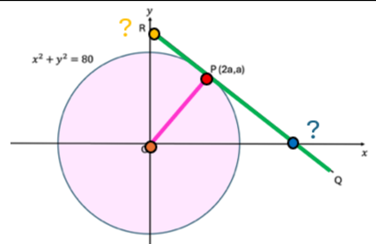
$$\frac{1}{2} \times 20 \times 6 = 60$$

GCSE AQA Churchill Practice Paper June 2017 Paper 1 Q23

A circle has the equation $x^2 + y^2 = 80$.



The centre of the circle is the origin, O. The point P on the circle has coordinates $(2a, a)$ where a is a positive constant. The tangent to the circle at P crosses the x axis at the point Q and crosses the Y axis at the point R. Work out the **area** of triangle OQR.



We need the tangent equation first before we can find R and Q

Find a by substituting point P$(2a, a)$ into the equation of the circle

$$(2a)^2 + a^2 = 80$$
$$4a^2 + a^2 = 80$$
$$a^2 = 16$$
$$a = 4 \text{ since } a \text{ is positive}$$

So P$(8, 4)$

Gradient of OP is $\frac{a-0}{2a-0} = \frac{1}{2}$

RQ is perpendicular to OP

$$\text{gradient of tangent at P} = -2$$

Find the full equation of RQ given that P$(8, 4)$ lies on the line

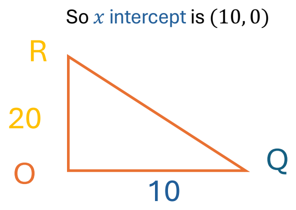
$$y = mx + c$$
$$4 = -2(8) + c$$
$$4 = -16 + 20$$
$$y = -2x + 20$$

R is the y intercept of RQ, which is $(0,20)$

Q is the x intercept of RQ

$$0 = -2x + 20$$
$$x = 10$$

So x intercept is $(10, 0)$

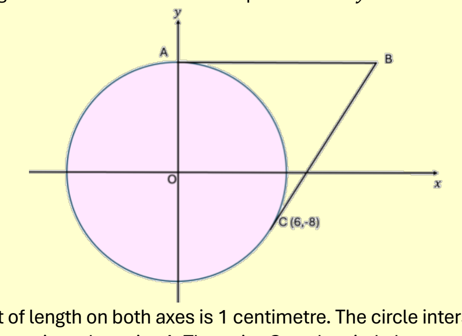


Find the area of the triangle

$$\frac{1}{2} \times 10 \times 20 = 100$$

GCSE Mock Set 4 Paper 1H Q21

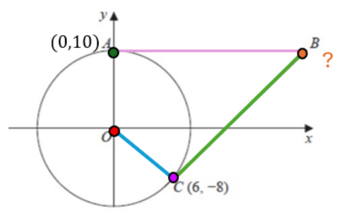
The diagram shows the circle with equation $x^2 + y^2 = 100$.



The unit of length on both axes is 1 centimetre. The circle intersects the positive y-axis at the point A. The point C on the circle has coordinates $(6, -8)$. The straight lines AB and CB are tangents to the circle. AB is a **horizontal line**

i.Find the **area** of quadrilateral $ABCO$

ii.Find the **area** of quadrilateral $ABCO$



We need both tangent equations first (tangent 1 and tangent 2) before we can find B

The radius of $x^2 + y^2 = 100$ is $\sqrt{100} = 10$

The gradient of OC is $\frac{-8-0}{6-0} = -\frac{4}{3}$

Tangent CB is perpendicular to OC:

$$\text{gradient of tangent at C} = \frac{3}{4}$$

Find the full equation of CB in the form $y = mx + c$ by substituting in the coordinates $(6, -8)$

$$-8 = \frac{3}{4}(6) + c$$
$$c = -\frac{25}{2}$$
$$y = \frac{3}{4}x - \frac{25}{2}$$

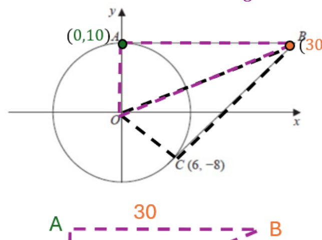
We know B is the intersection of AB and CB

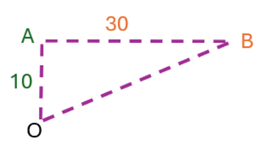
The line AB is just the equation $y = 10$. This will be the y coordinate of B

Equate AB and CB to find the x coordinate of B

$$10 = \frac{3}{4}x - \frac{25}{2}$$
$$\frac{3}{4}x = \frac{45}{2}$$
$$x = 30$$

$$Area\ ABCO = 2 \times triangle\ AOB$$





Base is length AO=10, Height is length AB=30

$$2 \times \frac{1}{2} \times 10 \times 30 = 300cm^2$$

Harder Examples – With Intersections

Intersections + Finding Circle Equation

The straight line L has equation $2x + y = 5$

C is a circle with centre the origin and radius 6

L and C intersect at the point A and B

i.Find the coordinates of A and B

ii.Find the equation of the tangent to the circle at B

$$2x + y = 5$$
$$x^2 + y^2 = 36$$

i.

$$y = -2x + 5$$
$$x^2 + (-2x + 5)^2 = 36$$
$$x^2 + 4x^2 - 20x + 25 = 36$$
$$5x^2 - 20x - 11 = 0$$
$$x = \frac{10 \pm \sqrt{155}}{5}$$
$$x = 4.49, -0.49$$

Find the corresponding y's

When $x = 4.49$:

$$y = -2(4.49) + 5 = -3.98$$

When $x = -0.49$:

$$y = -2(-0.49) + 5 = 5.98$$

$$A(4.49, -3.98)$$
$$B(-0.49, 5.98)$$

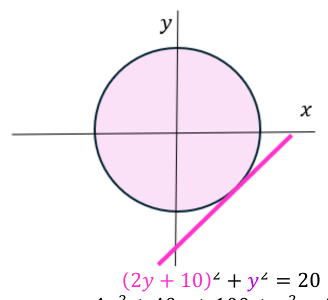
Intersections

GCSE Nov 2017 Paper 3H Q19

Prove algebraically that the straight line with equation $x - 2y = 10$ is tangent to the circle with equation $x^2 + y^2 = 20$

$x - 2y = 10$ can be written as $x = 2y + 10$

Substitute this expression for x into the equation of the circle $x^2 + y^2 = 20$



$$(2y + 10)^2 + y^2 = 20$$
$$4y^2 + 40y + 100 + y^2 = 20$$
$$5y^2 + 40y + 80 = 0$$
$$y^2 + 8y + 16 = 0$$
$$(y + 4)(y + 4) = 0$$
$$y = -4$$

Find x by substituting y value into $x = 2y + 10$

$$x = 2(-4) + 10 = 2$$

There is only one pair of solutions, so only one point of intersection, hence $x - 2y = 10$ is a tangent to the circle

Hardest Examples – With Other Topics

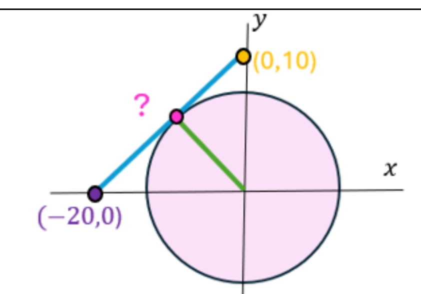
Finding The Circle Equation

GCSE Nov 2020 Paper 2H Q22

C is a circle with centre the origin.

A tangent to C passes through the points $(-20,0)$ and $(0,10)$

Work out the equation of C.



Need the pink point to work out the radius in order to find circle the equation.

We need the radius and tangent equation first to find the pink point

Find the gradient of the tangent

$$m = \frac{10-0}{0--20} = \frac{1}{2}$$

We can find the gradient of the normal by using the rule

The radius is perpendicular hence equation of the radius is $y = -2x$ because the centre is at the origin hence no y intercept

We already know the gradient of the tangent, and that it passes through a y intercept of $(0,10)$

The equation of the tangent is $y = \frac{1}{2}x + 10$

We have 2 equations and can substitute (1) into (2): to find x and y

$$y = -2x \quad (1)$$
$$y = \frac{1}{2}x + 10 \quad (2)$$
$$-2x = \frac{1}{2}x + 10$$
$$x = -4$$
$$y = -2(-4) = 8$$

Hence the coordinates of where the tangent and radius meet are $(-4, 8)$

The centre of the circle is $(0, 0)$ so the equation of C takes the form $x^2 + y^2 = r^2$

Find the radius

$$\sqrt{(-4)^2 + 8^2} = 4\sqrt{5}$$

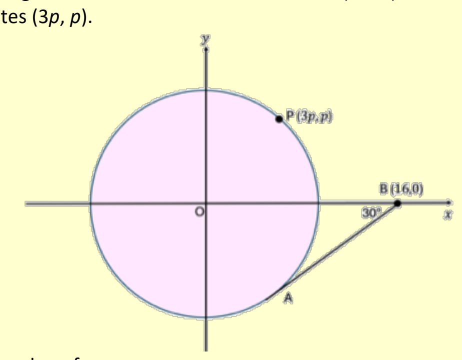
So, the equation of the circle is

$$x^2 + y^2 = 80$$

With SOHCAHTOA

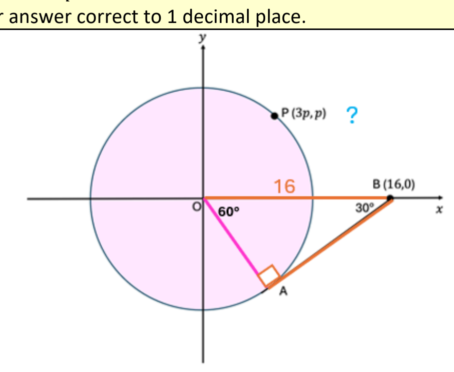
GCSE June 2019 Paper 3H Q22

The diagram shows a circle, centre O. AB is the tangent to the circle at the point A. Angle OBA 30°. Point B has coordinates (16, 0) Point P has coordinates $(3p, p)$.



i.Find the value of p.

ii.Give your answer correct to 1 decimal place.



Use the circle theorem that the angle between a radius (OA) and a tangent (AB) is a right angle

Find OA using the sine rule in the triangle

$$\frac{sinA}{a} = \frac{sinB}{b}$$
$$\frac{sinA}{OB} = \frac{sinB}{OA}$$
$$\frac{sin90}{16} = \frac{sin30}{OA}$$
$$\frac{1}{16} = \frac{\frac{1}{2}}{OA}$$
$$\therefore OA = 8$$

This means the radius is of length 8

Since the circle has centre (0,0), its equation takes the form $x^2 + y^2 = r^2$ and we know $r = 8$

So, the equation of the circle is

$$x^2 + y^2 = 64$$

Substitute the point P into this equation

$$x = 3p, y = p$$
$$(3p)^2 + p^2 = 64$$
$$9p^2 + p^2 = 64$$
$$10p^2 = 64$$
$$p^2 = 6.4$$
$$p = \pm\sqrt{6.4}$$

Since P is in the positive quadrant, we take the positive square root for p to 1 decimal place

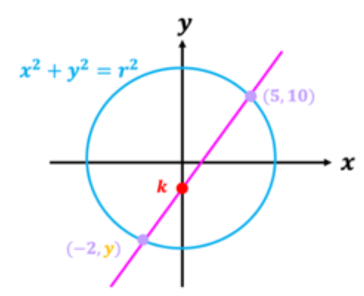
$$p = 2.5$$

With Circle Equation

GCSE June 2024 Paper 1H Q23

A straight line with positive gradient passes through two points on a circle centered at the origin. One of the points has coordinates $(5, 10)$ and the other has x-coordinate -2. Find the y-intercept of this line.

Firstly, draw out what is happening



To find k we need to set $x = 0$ in the equation of **the line** (or j locate the y intercept in the equation) but we don't have the equation of the line!

To find the line we need both points. To find the missing coordinate of point $(-2, y)$ we can plug it into the circle equation since it lies on the circle. But we don't have the circle equation either yet!

We know a point that lies on the circle so we can plug it in.

$$(5)^2 + (10)^2 = r^2$$
$$125 = r^2$$

Hence, the equation of the circle is

$$x^2 + y^2 = 125$$

So, we can plug $x = -2$ in now to find the y coordinate

$$(-2)^2 + y^2 = 125$$
$$y = \pm 11$$

The coordinate is below the y axis hence $y = -11$. Now we can use $(-2, -11)$ and $(5,10)$ to find the line equation.

$$m = \frac{10 - -11}{5 - -2} = \frac{21}{7} = 3$$

Hence

$$y = 3x - 5$$
$$k = -5$$